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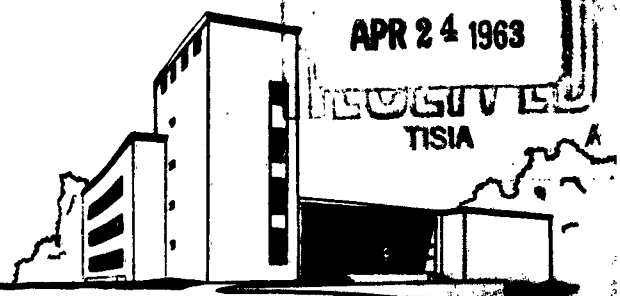

GRADUATE SCHOOL of INDUSTRIAL ADMINISTRATION

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SOLUTION OF A STOCKING PROBLEM BY A SHORTEST ROUTE ALGORITHM

by

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1. Introduction

Suppose some quality of a product varies according to some monotonic function of n numbers L_i, which we shall call for convenience lengths. The problem is to determine which of these n lengths to stock (henceforward called standards), given that we can stock only $m<n$ different lengths, such that the loss incurred in reducing or cutting or supplying standards to the demanded lengths is minimized. It is assumed that:

A. The part of the product that is left after reduing, i.e., the scrap, cannot be used again.

B. The loss is a function of the scrap only.

C. All the lengths are demanded equally.

Assumptions B and C are assumed to hold only in Sections 1 and 2. For instance if we are given three lengths of steel beams of 4 ft., 3 ft., and 1 ft. at a cost of $1 a ft. so that the loss in reducing L_i to L_j is just the cost of L_i-L_j, and told to choose two standards, we will pick 4 ft. and 1 ft. as our lengths. To see this, note that 4 ft. must be chosen as a standard, since it is the longest, must be supplied, and anything longer is just wasted (if other than demanded lengths were permitted as standards). For the other standard either 3 ft. or 1 ft. must be chosen, since again, not choosing a demanded length as a standard just wastes the difference between its length and the greatest demanded length less than it, at least. Now it can be seen that 1 ft. must be chosen as a standard since its waste is only $1 (= $4 - $3) and the waste with 3 ft. is $2 (= $3 - $1). As another example suppose we had square box tops with sides of lengths 4 ft., 3 ft., and 1 ft. demanded and the cost is $1 a square foot, and we could stock but two standard lengths. The loss in

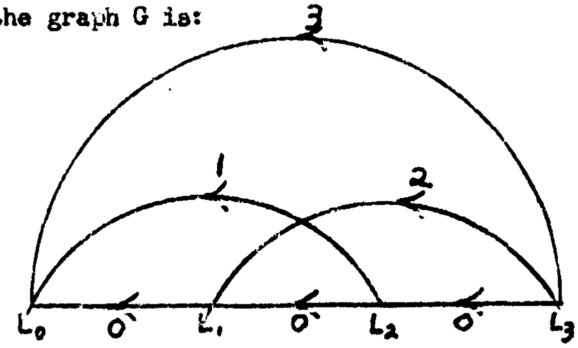
not stocking the 1 ft. top would be $\$3^2 - \$1^2 = \$8$, and the loss in not stocking the 3 ft. top would be $\$4^2 - \$3^2 = \$7$. Therefore we would stock the 4 ft. and 1 ft. box tops.

2. The Solution

With the given lengths L_i we associate the transitive, complete, asymmetric graph G, to be defined. The graph G will also be irreflexive in this section but not in Sections 3 and 4. Each vertex in G corresponds to one of the L_i, and one more vertex is added to serve as a sink. The vertex corresponding to the longest length, L_n, is the source. Assume the demanded lengths are ordered, i.e., $L_i \leq L_{i+1}$ for all $i<n$. Call the added vertex (the sink) L_o. Now draw an arc from L_i to L_j if $i>j$ and associate with this arc the loss incurred in assuming that L_i and L_j are standards and that there are no standards between them, and counting only that part of the loss incurred in supplying the lengths intermediate between L_i and L_j. Arcs from L_i to L_o count the loss incurred in supplying L_i in assuming that L_i is the smallest standard. Thus the arc from L_i to L_j,

$$[L_i, L_j] = \sum_{n=j+1}^{i-1} f(L_i, L_n)$$

where $f(L_i,L_n)$ is the loss from L_i in supplying L_n. Using the steel beam example the graph G is:



Now let u_m be a set of m of the L_i, and $C(u_m)$ be the cost incurred in assuming that each of the u_m are all standards and A, B, and C. Also let u'_m be the L_i obtained from G, defined as above, which correspond to the shortest m step path from L_n to L_o. Let $L(u_m)$ be the length of the path in G corresponding to u_m. Then,

Theorem: Given $L_1 \le L_2 \ldots \le L_n$ and $m<n$, and assuming A, B, and C, then

a. $C(u_m) = L(u_m)$ and hence

b. $\min_{u_m} C(u_m) = C(u'_m)$

where the minimum is taken over all u_m.

Proof: It is clear from the nature of G that each u_m corresponds to a unique m step path from L_n to L_o, and that $C(u_m)$ is the length of that path since the length of each arc is the loss incurred from each standard assuming A, B, and C. Therefore u'_m incurs the least cost and $\min_{u_m} C(u_m) = C(u'_m)$.

There are several ways of finding the shortest m step path from L_n to L_o. In [4] the shortest path is found using matrices, and it can be seen to be extendable to the shortest m step path. Another more efficient method is to label $L_n = 0$ and all the other $L_i = \infty$, then relabel each L_j m times starting with L_o and taking them in order each time. The label on L_i each time is $\min_{j>i} (L_j + [L_j, L_i])$ where $[L_j, L_i]$ as before is the length of the arc from L_j to L_i. This is equivalent to remembering only the nth row or column from A^2 to A^m where A is the matrix associated with G as described [1]. Now in order to find the path we at the kth step associate with each vertex L_i the k step path used to obtain the labels of L_i. When L_i is being relabeled j and the path used to get to L_j is the new path associated with L_i, where j is determined by the minimum function. It can also be seen that we need only use $n-m$ locations at

a time for remembering paths since the path associated with L_k on the kth step is just $L_n,\ldots,L_k$ and since the path associated with L_j for $j<m-k$ need not be remembered since it will never be used or needed. Similarly only $n-m$ locations need be remembered for the labels of the L_i since as before we calculate the label for $L_{m-1},\ldots,L_{n-2}$, then $L_{m-2},\ldots,L_{n-3}$ etc., until L_m is labeled.

3. Generalization

Now let us relax assumptions B and C. In their stead we will say that there are given probabilities of demand, p_i, for each L_i. We also have a function $c(L_i,L_j)$ of the endpoints that could correspond to inventory or space costs of keeping a supply of standard lengths, L_i, of the product in order to meet the expected demand for lengths greater than L_j and less than or equal to L_i during a period of time, where L_i and L_j are assumed to be standards with no standards between them. It could also correspond to the initial cost of building a container for the standard length L_i. Thus in G let $[L_i,L_j] = c(L_i,L_j) + \sum_{n=j+1}^{i-1} p_n f(L_i,L_n)$. Now using this graph we generalize a number of ways. First, if the problem, as before, requires that we use exactly m standards, we let $[L_i,L_i] = \infty$ and $C(u_m) = L(u_m)$ and $\min_{u_m} C(u_m) = C(u_m')$ using the notation of the Theorem. Second, if we require the use of m or less standards, i.e., $\min_{\substack{k\leq m\\ u_k}} C(u_k)$ we then let $[L_i,L_i] = 0$ in G and use the same algorithm as before for the m step path only permitting these loops to be used. Then if u_m' denotes the shortest m or less step path, $\min_{\substack{k\leq m\\ u_k}} C(u_k) = C(u_m')$.

Third, if we rid ourselves entirely of m and just require that the cost be minimized regardless of the number of standards, $[L_i,L_i] = 0$ and the

last equation becomes $\min_u C(u) = C(u')$ where the minimum is taken over all u and u' is the shortest path and may be found by any of the standard methods. Fourth, if space or inventory costs are due to just the number of standards regardless of which of the L_i are standards, we let $[L_i, L_j] = \infty$ again in G and calculate $\min_{u_m} C(u_m)$ for each m. Then $C(u) = \min_m (C(u'_m) + K(m))$ where u is the set of standards incurring the least total cost.

4. Example

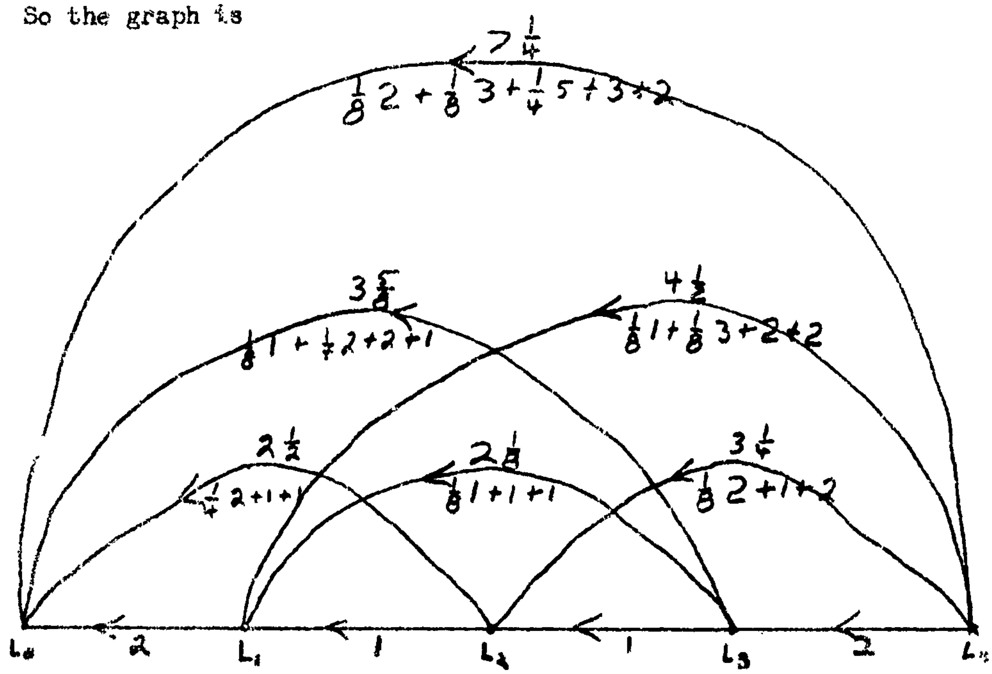
Let $L_1 = 1$ $\quad p_1 = 1/4$ $\quad k_1 = 2$ $\quad$ and $c(L_i, L_j) = i - j - 1 + k_i$

$L_2 = 3$ $\quad p_2 = 1/8$ $\quad k_2 = 1$ $\quad$ $f(L_i, L_j) = L_i - L_j$

$L_3 = 4$ $\quad p_3 = 1/8$ $\quad k_3 = 1$

$L_4 = 6$ $\quad p_4 = 1/2$ $\quad k_4 = 2$

So the graph is



Therefore the shortest path is L_4, L_3, L_2, L_0 and the cost is 5 1/2.

References

1. Berge, Claude, The Theory of Graphs and its Applications, London: Methuen and Co. Ltd., New York: John Wiley and Sons, Inc. 1962, p. 139.

2. Dantzig, George B., "On the Shortest Route Through a Network," Management Science, Vol. 6, No. 2, January, 1962.

3. Dantzig, George B., "Discret-Variable Extreme Problem," Opns. Res. 5, 266-270 (1957).

4. Peart, Robert M., Paul H. Randolph, and T. E. Bartlett, "The Shortest-Route Problem," Opns. Res. 8, 866-868 (1960).

5. Polluck, Maurice and Walter Wiebenso, "Solutions of the Shortest-Route Problem -- A Review," Opns. Res. 8, 224-230 (1960).